Γ

Mister

« *Lady Gamma* »

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The book still being updated, any mistakes occurring here are mine.

Preface

The *factorial function* known today under the name of *Gamma function* and the notation $\Gamma(x)$, is due to the genius of mathematician and physicist Leonard **Euler**(1707-1783). It has been a subject of research since the XVIII century by some of the greatest mathematicians.

To the following question:

What is the function f *of real variable* x *that has the propriety:* $f(n) = n!$?

Euler found two solutions

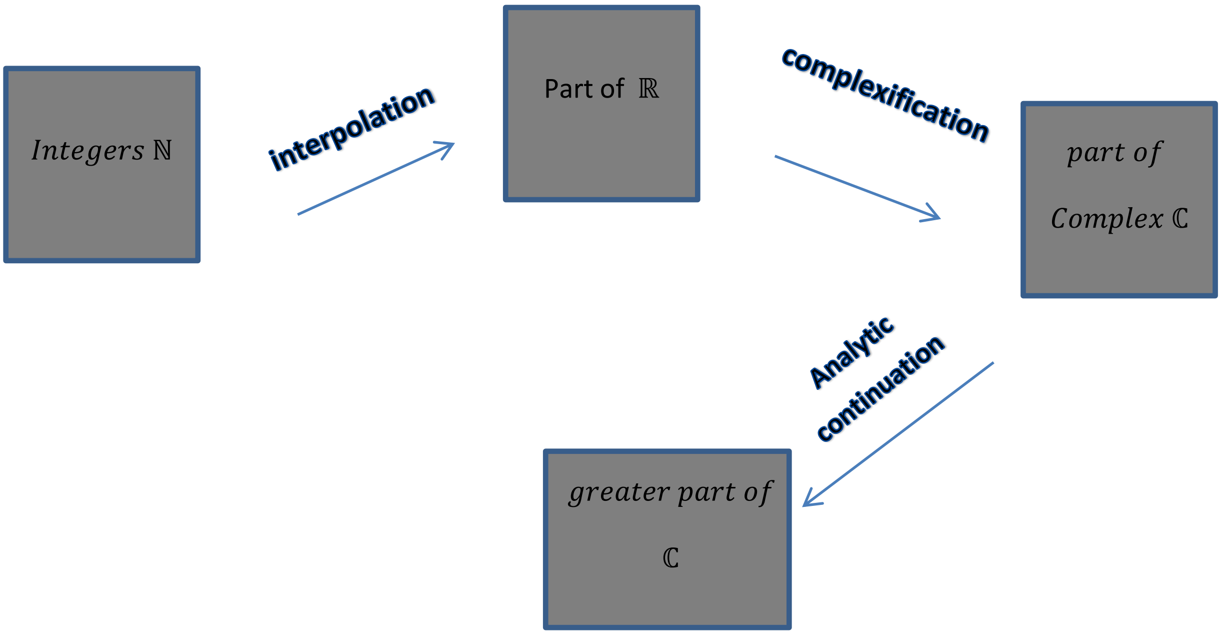
— the first solution consist in that f can be given as an infinite product of terms $\Gamma(x) = \prod_0^\infty a_i$.

— and the second solution is that f can be represented as a definite integral $\Gamma(x), = \int_a^b g(x)dx$.

And since that, as we will see, and from the point of view of *rigor*, there has been no agreement on what should be the right definition. The reader will learn thorough these notes that there have been many points of view to define the celebrated function, and in a lot of references he will read its two definitions among statements like: *the gamma function is the only function that verify such or such….*

Anyway, in modern terminology, *Gamma function* belongs to a large set of complicated functions used in many branches of modern science, formerly named *transcendentals*. Nowadays these functions are called *special functions* , very famous and useful functions like : theta function $\vartheta(z)$, **Bessel** functions , hypergeometric functions, **Airy** function, **Hermite** function….
Clearly, by its origin, the Gamma function belongs to the world of mathematical analysis, but it is very important in many other mathematical contexts, either in pure or in applied mathematics. In pure mathematics for example, it appears in the theory of numbers, where it plays a preponderant role in the comprehension of the zeta functions linked to the distribution primes, and also in probability, and in theoretical physics… Regarding applied mathematics, it also appears in many domains like: statistics, applied functional analysis, differential equations…

The gamma function was first got in the context of real variables, then as many other important functions, has been considered for complex variables. In the case of gamma, the transition from $\mathbb{R}$ to complex analysis in $\mathbb{C}$ was first accomplished by **Gauss** then used much by **Riemann** in his big work on the zeta function. So the image of the process around the transition from real to complex functions can be in the chart bellow:



The following notes, were originally annexed to my essay on the **Riemann** zeta function, and I have decided to expand and put it in an independent part, and I preserved the original title

Mister « *Lady Gamma* ».

I wanted these notes to contain some comprehensive information, both technical and historical, on this function presented in an easy manner to beginners.

In fact the three main preoccupations of scientists that led to the calculus and other discoveries in analysis like our Gamma were:

*Quadrature of the circle.

*Finding areas and volumes.

*Triangular numbers.

So I will begin form the early works on the operation of *interpolation* which has led to the discovery of the gamma function, and the work done after the great **Euler** wrote its expression down.

At the end of each part of these notes, I added some exercises. The exercises are related to the subject of each chapter; they are on calculus and its standard problems that turn around: convergence, calculation of definite and indefinite integrals, complex variables… These drills are meant to hint to the properties of this function in a way or in another, so I hope also that they help the reader to understand more this function.

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Finite differences

Before the definitive invention of infinitesimal calculus and its powerful methods, scientists used tricks to find values of functions by the operation of *interpolation* in the form of finite differences, a technic very fruitful and linked to Calculus as Georges **Boole** said in the beginning of his book

> *In the following exposition of the calculus of finite differences, particular attention has been paid to the connection of its methods with those of the differential calculus, a connection which in some instances involves far more than a merely formal analogy.*

And further he defined it

> *The calculus of finite differences may be strictly defined as the science which is occupied about the ratios of the simultaneous increments of quantities mutually dependent. The differential calculus is occupied about the limits to which such ratios approach as the increments are indefinitely diminished.*

What we known about this in common references is that interpolation existed in the work of Jacob **Bernoulli** (1655-1705), seen as the early typical example of interpolation of sequences but this mathematical philosophy of thinking about numbers was not new at the time, there have been some earlier extensive work that treated interpolation . In fact, before **Bernoulli** was working on number combinations for probability theory, the language of *interpolation* was much used by the scientists of the XVI century.

So the beginning of our story about the gamma function can be found in an activity much done by mathematicians, especially in the XVII century, an consisting in using clever formulas to *generalize* sequences of numbers, and this in general can be accomplished by two methods:

1) *Interpolation*. The operation of finding values governed by a law within an interval between two real numbers a and b.

2) *Extrapolation*. The operation of finding values, governed by a law, outside the interval bounded by two real numbers a and b.

Simple examples of interpolation

Some interpolation operations can very simple, for example to generate all odd numbers we may calculate the differences of the squares like this:

0 1 4 9 16 25 36 49 64
1 3 5 7 9 11 13 15

During the early period known before steps to the invention of calculus, with for example the triangular numbers

1, 3, 6, 10, 15, 21, 28, 36, 45,…

Which known to exist from **Pascal**[1] triangle

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[1] This triangle have been discovered first by Muhammed **Alkaraji**(ca. 1050) then by **Tshu Shi** (1303)and last by **Stifel** (1544).

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

The triangle shows three numbers: *triangular* numbers (in red), *binomial* number (horizontal numbers beginning by the 1s) and *pyramidal* numbers (in blew). The ***Pascal****'s law* for difference

$$\binom{n}{k} = \binom{n+1}{k} - \binom{n}{k-1}$$

Hariot and the others

Many mathematicians of the XVII century like **Napier** (1-1) , **Mercator**(1-1) and **Bürgi**(1552-1633)…were implied in practical calculations for navigation and other purposes. In fact the work done by those men can be seen today as a part of *numerical analysis*.

Such mathematicians used interpolation in constructing:

-Tables of sins for angles.

-Tables of tangents.

-Tables of logarithms, newly invented by **Napier** and **Bürgi**

-Tables of antilogarithms.

English scientist Thomas **Harriot**[2] (1560-1621) was an early English mathematician of the XVI century who influenced a lot of English mathematicians like **Wallis** and **Newton**.. He founded his work upon triangular numbers and to the laws governing the binomials before **Newton**

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[2] He has used the equal sign "ㅍ" similar to **Record**'s " = " and we owe him the two signs < and >.

$$1, \quad \frac{n}{1}, \quad \frac{n(n-1)}{1.2}, \quad \frac{n(n-1)(n-2)}{1.2.3}, \dots$$

The aim of **Hariot** was practical; he was implied in such mathematical activity mainly for the construction of navigational tables.

His work consists in interpolating natural integers like those in arithmetic progressions by the study of constant finite differences

$$\Delta^1 y = \Delta y = y_{k+1} - y_k$$

and other higher differences

$$\Delta^n$$

as the example that precede. We can say here that as the derivate f' gives us information about f, also the difference Δ^n can led us to some valuable information on the sequence.

Wallis

Wallis (1616-1703) is also known for using interpolation and his work, we will see, his formula for $\frac{4}{\pi}$ inspired **Euler** in his choosing the integral representation of the gamma function. **Wallis** was obsessed also by two problems:

* of finding areas of figure (called *quadrature*).

*and volumes(called *cubature*).

In his book *Arithmetica infinitorum*, he was concerned much of the search for the means of arithmetic progressions. He for example found that the *means of the sequence of integers* to the sum of last term is $\frac{1}{2}$

$$\frac{1+2+3+\cdots+n}{n+n+n+\cdots+n}=\frac{1}{2}$$

an easy limit by the triangular number general formula. Some of the sequnces was hard, and he particularly asked help from the mathematicians of Oxford for the mean of the sequence n!

$$1, \quad 6, \quad 30, \quad 140, \quad 630\,, \ldots$$

after futile attempts he concluded that such law governing the mean can neither be arithmetic nor geometric. In fact, he commented (p. 161)

> *…and indeed I am inclined to believe (what from the beginning I suspected)that this ration we seek in such that it cannot be forced out in numbers according to any method of notation so far accepted, not even by surds… so that it seems necessary to introduce another method of explaining a ration of this kind, than by true numbers or even by the accepted means of surds.*

Wallis was interested in sums of sequences of powers

$$a, \quad a^2, \quad a^3, \quad a^4, \ldots$$

sums linked to quadrature of the form

$$(1 - x^{\frac{1}{p}})^q$$

One problem especially occupied him and as well as generations of geometers before, it was the famous *squaring the circle* inherited from the ancient Greek. The search was the reason that led him to the quadrature of

$$\sqrt{1 - x^2}$$

And thus to a ratio for the magic number π

$$\frac{2 \times 2 \times 4 \times 4 \times 6 \times 6 \times 8 \times \dots}{3 \times 3 \times 5 \times 5 \times 7 \times 7 \times \dots} = \frac{\pi}{4}$$

we see that this fraction has split $n!$ in two parts : odd numbers in the denominator and even ones in the numerator .

Trying to find such a relation in modern notation

$$\int_{-1}^{1} (1 - x^2)^{\frac{1}{2}} dx = \pi$$

He was implied in calculating definite integrals

$$\int_{0}^{\pi/2} sin^2(x) dx$$

and with the exchange $t = cos(x)$ we can find

$$\int_0^1 (1-t)^{(n-1)/2} t^{-1/2} dx$$

and we will see that a similar in $0 < x < 1$ to

$$\int_0^1 (1-t)^{-x} t^{x-1} dt = \frac{\pi}{sin(\pi x)}$$

a known identity for the gamma function.

Newton

The work of **Wallis** has influenced **Newton**(1643-1727) and led him to the binomial theorem. In fact before the posthumous work of **Bernoulli**, **Newton** himself inspired by **Wallis** and in 1665 , *interpolated* the very known *binomial theorem* for integers

$$(a+b)^n = \sum_{k=1}^{n} \binom{n}{k} a^n\, b^{n-k}$$

and he also went further to consider in place of the integer n the rational numbers x, but although $\binom{n}{k}$ has by definition no meaning in this case when n is rational, the operation of subtraction has a meaning

$$(1+b)^x = 1 + \frac{x}{1}b + \frac{x(x-1)}{1.2}b^2 + \frac{x(x-1)(x-2)}{1.2.3}b^3 + \cdots$$

all this was published **Newton**'s *Methodus Differentialis* in 1711. **Euler** later used the same formula when n tends to $+\infty$, to find the value of e

$$e = \left(1 + \frac{1}{n}\right)^n = 1 + \frac{n}{n} + \frac{n(n-1)}{1.2}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{1.2.3}\frac{1}{n^3} + \cdots$$

So we see here the appearance of the sequence $n!$ in the terms' denominators.

Another problem seen also as an *interpolation*, is to find a polynomial $P_n(x)$ that passes by some finite number of points : suppose we have an unknown function or law f that takes the values

$$y_1, \dots, y_n$$

at the n points

$$x_1, \dots, x_n$$

as polynomials are very good functions for their simplicity of calculation, we desire to find a polynomial that passes through all of them, and thus approximates the unknown f at those points. Two methods were discovered and have been used to solve this problem and both used finite differences, the first was by **Newton**(1676) :

$$P(x) = y_0 + \sum_1^n \frac{}{n!}\frac{(x-x_n)}{!\,h}\Delta^n y_i$$

Where $h = x - x_0$. And the second by **Lagrange**(17) and his interpolation polynomial has the form

$$P(x) = \sum_{i=1}^{n} \frac{(x - x_{i+1}) \dots (x - x_n)}{(x - x_{i-1}) \dots (x - x_n)} y_i$$

There exists also in this domain interpolations that use rational and trigonometric functions in place of polynomials, developed later by **Cauchy**. The use of this technics is not at all old fashioned, it is still used in numerical analysis and since the XVII century many mathematicians used them in many important situations to solve problems , in 1860 **Boole** wrote a big book of more than 400 pages on this subject, still in use.

Exercises

1) make the differences of first ten consecutive cubes, what do you remark?

1) show that : $\binom{n}{k} = \frac{n \times (n-1) \times \ldots \times (n-k+1)}{k!}$

1) prove ***Pascal**'s law* as a difference

$$\binom{n}{k} = \binom{n+1}{k} - \binom{n}{k-1}$$

1) Show that $2^n = \sum_{k=1}^{n} \binom{n}{k}$. This number is the number of all possible combinations of n elements, and the number of all subsets of a finite set E.

1) Let $(1,2), (-1,3)$ be two points in the plane $\mathbb{R}^2$. Find the polynomial of degree 2 that passes through these two points.

1) prove the known triangular number formula:

$$f(n) = t_n = \sum_{1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n}{2}(n+1)$$

1) show that the limit as $n \to +\infty$

$$\frac{1 + 2 + 3 + \cdots + n}{n + n + n + \cdots + n} = \frac{1}{2}$$

1) let the integral of the form

$$\int_{\alpha}^{\beta} sin^p(x)cos^q(x)dt$$

where at least one of the p, q is odd. Find suitable changes that lead to the integral

$$\int_{\alpha}^{\beta} (1 - t^2)^k t^p dt$$

The Bernoullis

In the XVII century, mathematician Jacob **Bernoulli** (1655-1705) in jotting with numbers, was interested to interpolate sums of the form

$$f(n) = \sum_{1}^{n} k^{r}$$

and find closed formula for them for given integer n and r , which means that we have only to put $n = n_0$ to get the searched sum . A special case of this formula is that of the known triangular numbers:

$$\sum_{1}^{n} k^{1} = 1 + 2 + 3 + \cdots + n = \frac{n}{2}(n+1)$$

where in this case $r = 1$, as we have seen.

Another known formula when $r = 2$

$$1^2 + 2^2 + 3^2 + \cdots + {n_0}^2 = \frac{n_0(n_0 + 1)(2n_0 + 1)}{6}$$

if we think differently , **Bernoulli** was seeking a law $f(n)$ for the sequence

$$1^r, \quad (1 + 2^r), \qquad (1 + 2^r + 3^r), \ldots$$

Thus doing this, he discovered the very famous ***Bernoulli*** *numbers*, and the law which Jacob **Bernoulli** looked for is :

$$\sum_{1}^{n} r^k = (-1)^b \binom{1}{n+1} \frac{B_n}{}$$

The ideas of **Bernoulli** on integers and combinations of numbers , with his formulas were published posthumously in a book entitled *Ars Conjuctiri* (1713), which means *art of conjectures*, the book dealt about probability.

Goldbach

During the following century, i.e. the XVIII century, the son of Jacob , Daniel **Bernoulli** (1700-1782) , and German Christian **Goldbach** (1700-1782) were occupied by the same problem and this operation was known to be called by the same name *interpolation* . So if we succeed in finding this such function we first :facilitate to give the values of this sequence for any natural numbers and avoid possible inextricable calculations, and second : we interpolate the sequence to other argumentss that are not integers, and thus we may discover a new useful general function perhaps more important and promising than our sequence .

Both the two mathematicians were searching this time for two functions:

i) the first f interpolating the *harmonic* sequence u_n

$$1+\frac{1}{2}, \qquad 1+\frac{1}{2}, \qquad 1+\frac{1}{2}+\frac{1}{3}, \qquad \ldots$$

Which means: find a function sum

$$f(n) = 1+\frac{1}{2}+\cdots+\frac{1}{n}$$

ii) the second g that give the hypergeometric sequence

$$g(n) \ = \ n\,!$$

For every natural integer n, so in this case the task was to *interpolate* the *sequence(or progression)* defined by the product

$$n! = \prod_{0\le k<n} (n-k)$$

So in the case of the factorial , or more promising task, the formula that was originally valid only for integer numbers n , may be generalized and have values when the argument n (or x in $f(x)$) is not necessarily an integer :

$$f(x) \ = \ x\,!$$

Or said otherwise, the task was to find a formula in function of n that easily gives the searched term of the sequence

$$u_n = n\,!$$

without calculating it by other means, a task which **Euler**'s called in his prospective paper : " the most important problem in the subject of sequences"[3].

Exercises

1) calculte

$$\sum_{1}^{10} k^1 \ , \quad \sum_{1}^{10} k^2 \quad , \quad \sum_{1}^{10} k^3$$

1) in the spirit of Jacob **Bernoulli**, show that :

$$\sum_{1}^{n} r^3 = \left[\frac{n(n+1)}{2}\right]^2$$

1) it is said that **Gauss** when he was a pupil calculated the sum $\sum_1^n k^1$ =by writing :

$$S_n = 1 + 2 + 3 + \cdots + (n-1) + n$$

$$S_n = n + (n-1) + \cdots + 1$$

complete the calculations to find the sum : $S_n = \frac{n}{2}(n+1)$.

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[3] The summary of the paper.

1) Show that $2^n = \sum_{k=1}^{n} \binom{n}{k}$. This number is the number of all possible combinations , i.e. the number of subset of a finite set.

1) show that

$$\int_0^{+\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$$

Euler I

When **Euler** entered the scene in 1728, he has just left Germany for Russia and heard of Daniel **Bernoulli** and **Goldbach** researches.

He saw the problem very important, in fact in his coming paper(1831) we can read the following

> *The theory of progression of numbers, not merely because of its beauty, but also of its great utility deserves the attention of Geometers[4]. The most important in this subject is to exhibit, for any given progression, its general term, i.e., that general formula or expression from which any desired term of the progression could be found separately and immediately, apart from the other terms.*

Euler, asking the same question: find a law f that gives the terms of the sequence $u_n = n!$:

$$1, \qquad 2, \qquad 6, \qquad 24, \qquad 120, \ldots$$

Or find a function f such that :

$$f(n) = u_n = n!$$

---

[4] He means "*mathematicians*".

He solved the problem in two papers, and to be we can retrace the events before the:

$*$) **Euler** arrived at Russia in 1728 and began afterwards to think about the problem.

$*$) He first found that the searched function $f(n) = n!$ cannot be algebraic but transcendental.

$*$) He corresponded with **Goldbach** about his thoughts on the 3rd of October 1729

, and announced that he discovered a product for this function $f(n)$.

$*$) in another letter to **Goldbach** , dated on 8 of June 1830, he presented the gamma function as a definite integral.

Algebraic and transcendental

When **Euler** began tackling the problem(only 23 years old), he first observed :

Now there are some progressions, the general terms of which either cannot be exhibited at all, or at least not by any algebraic expression. Hence the author considers progressions of this type, the general terms of which cannot be given by any algebraic formula. Thus, e.g., one might consider the progression

1 , 2 , 6 , 24 , 120 ,...

whose general term cannot be exhibited algebraically,

So he formulated the problem otherwise: he classified progressions in two categories : algebraic and transcendental. **Euler** supposed that sequence $n!$ perhaps may be found by either algebraic or at least exponentially, then he changed his mind as we will see later on.

i) *algebraic* : a progression that we can directly get the values from its general by finite number of operations like the triangular number t_n

$$t_n = 1 + 2 + 3 + \cdots + n = \frac{n}{2}(n+1)$$

In this case for example, in order to find a term of the sequence $\sum_1^n k$

$$1, \qquad 3, \qquad 6, \qquad 10, \ldots$$

it suffices to put value of n in $\frac{n}{2}(n+1)$. And there is no problem in saying that the function that gives the sum of the first n integers is

$$f(x) = \frac{x(x+1)}{2}$$

ii) *Transcendental*: a progression that is *not algebraic* , that has usually infinitely many terms. Which **Euler** named *transcendental*, he means a progression which *transcends* **(***just as everything in geometry which surpasses the powers of ordinary algebra is commonly transcendental* (sic)**)** the easy nature of algebraic sequences and functions.
He stated that the progression

$$n!$$

is difficult in nature and it is impossible to find an algebraic formula, i.e. finite like that giving any triangular number t_n, for it thus it is impossible by any algebraic expression.

So , after the suggestion of Daniel **Bernoulli**, **Euler** communicated his early thoughts in a letter to **Goldbach** (1690-1764) dated on the 3rd of October, 1729, **Euler** announced that he has solved the problem by putting it in a product.

He solved the other problem proposed by **Bernoulli** and **Goldbach** namely a closed formula for the sequence

$$u_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

and he found also a very interesting approximation

$$\lim_{n \to +\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - log(n) \right) = \gamma = 0.577 \ldots$$

This is also named gamma and is very obscure number, we do not even if it is rational[5]. Curiously the two problems are related through the equality

$$\frac{1}{x!} = xe^{\gamma x} \prod_{k=0}^{\infty} (1 + \frac{x}{k})\, e^{-x/k}$$

as we will see.

The number $n!$ as a product

After this correspondence with **Goldbach**, **Euler** published an article (1730) in Latin untitled " *De progressionibus transcendentibus seu quarum termini generales algebraice dari nequeunt* " , which is in English[6]

ON TRANSCENDENTAL PROGRESSIONS
THAT IS, THOSE WHOSE GENERAL TERMS
CANNOT BE GIVEN ALGEBRAICALLY

and in another letter dated on 8 of January the same year 1730 he announced that he could present the function as an integral.

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[5] See the book by Julian **Havil** : *Gamma : exploring Euler's constant*, published in 2003.

[6] Translated into English by Stacy G. **Langton** (1999).

Euler begins by saying that the progression $u_n = n!$ at infinity became like a geometric progression, this is because $n!$ looks like

$$n! = 2^{k_1} \times 3^{k_2} \times 5^{k_3} \times 7^{k_4} \times \dots$$

He had the idea to introduce the geometric progression r^n , he wrote the general term of $u_n = n!$ as an infinite product:

$$u_n = \frac{1.2^n}{1+n} \times \frac{2^{1-n}.3^n}{2+n} \times \frac{3^{1-n}.4^n}{3+n} \times \frac{4^{1-n}.5^n}{4+n} \times \dots$$

in fact this is simply equals by elimination of numbers to the product

$$u_n = \frac{1}{1+n} \times \frac{2}{2+n} \times \frac{3}{3+n} \times \frac{4}{4+n} \times \dots$$

or

$$\boldsymbol{n}! = \frac{\boldsymbol{k}!}{\prod_{\boldsymbol{k}=1}^{+\infty}(\boldsymbol{k}+\boldsymbol{n})}$$

So we have

$$u_0 = 0! = \frac{1}{1+0} \times \frac{2}{2+0} \times \frac{3}{3+0} \times \frac{4}{4+0} \times \dots = 1$$

$$u_1 = 1! = \frac{1}{1+1} \times \frac{2}{2+1} \times \frac{3}{3+1} \times \frac{4}{4+1} \times \dots = 1$$

$$u_2 = 2! = \frac{1}{1+2} \times \frac{2}{2+2} \times \frac{3}{3+2} \times \frac{4}{4+2} \times \dots = 2$$

$$u_3 = 3! = \frac{1}{1+3} \times \frac{2}{2+3} \times \frac{3}{3+3} \times \frac{4}{4+3} \times \dots = 6$$

The reader with knowledge of the infinite process, sees at once the problem of convergence of this infinite product. But **Euler** as it is known about him in other works, does not bother himself usually about the convergence of such infinite series and products, and may writes something like:

$$1+\frac{1}{2}+\frac{1}{3}+\cdots=0$$

An infinite product in modern terminology and notation can be written

$$u_n=\prod_{1}^{+\infty} v_n=\lim_{n\to+\infty}(v_1\times v_2\times\ldots\times v_n)$$

In order that an infinite series

$$\sum_{1}^{+\infty} u_n$$

converges, a necessary (but no sufficient) condition is that :

$$\lim_{n\to+\infty} u_n=0$$

For infinite product, also a necessary but not sufficient condition for convergence is that

$$\lim_{n\to+\infty} u_n=1$$

the product

$$u_n = \frac{1}{1+n} \times \frac{2}{2+n} \times \frac{3}{3+n} \times \frac{4}{4+n} \times \dots$$

is not defined when n is a negative integer. We, like **Euler** did**,** try to put some integers in the argument to find out the results:

$$n = 0, 1, 2, 3.$$

Then for $n = 0$ and $n = 1$

$$0! = 1\,! = \frac{1.2^0}{1+0} \dots = \frac{1.2^1}{1+1} \dots = 1$$

for $n = 2$

$$2\,! = \frac{1.2^2}{1+2} \times \frac{2^{-1}.3^2}{2+2} \times \dots = 2\frac{2}{1.3} \times \frac{3.3}{2.4} \times \frac{4.4}{3.5} \dots = 2$$

and also $n = 3$

$$3\,! = \frac{1.2^3}{1+3} \times \frac{2^{-2}.3^3}{2+3} \times \dots = 2.3\frac{2^2}{4} \times \frac{3^2}{4.5} \times \frac{4^3}{3^2.6} \times \frac{5^3}{4^2.7} \dots \dots = 6$$

…

one must realize that the products are infinite, but **Euler** says that it always converge for real numbers(except for negative integers). Otherwise if $n = 0$, we have:

$$\frac{1}{1}\times\frac{2}{2}\times\frac{3}{3}\times\frac{4}{4}\times\ldots=1$$

If $n=1$, we have :

$$\frac{2}{2}\times\frac{3}{3}\times\frac{4}{4}\times\frac{5}{5}\times\ldots=1$$

So at the time we consider the product as a finite product, and **Euler** added about this product

This expression, indeed, never breaks off whether n is a whole number or a fraction, but only gives approximations for any desired term, except in the cases $n = 0$ and $n = 1$, in which it just becomes 1.

So he consider as an interpolation and not only an expression for integers $n \in \mathbb{N}$ but also when $x \in \mathbb{Q}$.

$$x\,! = \frac{1.2^{x}}{1+x}\times\frac{2^{-2}.3^{x}}{2+x}\times\ldots$$

whose meaning will be apparent later on.

Exercises

1) are the following products converging:

i) $\prod_1^{+\infty}(1+\frac{1}{n})$

ii) $\prod_1^{+\infty}(1-\frac{1}{n^2})$

1) how can treat the infinite product

$$v_n = \frac{1.2^n}{1+n} \times \frac{2^{1-n}.3^n}{2+n} \times \frac{3^{1-n}.4^n}{3+n} \times \frac{4^{1-n}.5^n}{4+n} \times \dots$$

Euler II

The appearance of the number π

Euler went on by saying that the previous expression for $n!$ i.e. the product

$$x\,! = \frac{1.\,2^{x}}{1+x} \times \frac{2^{-2}.\,3^{x}}{2+x} \times \dots$$

was not at all practical, or as he puts it "*of any use for computing the terms…*" :

> *However, although this expression would not appear to be of any use for computing the terms of the series, nevertheless it is wonderfully suitable for interpolating terms whose indices are fractional numbers. I have decided, however, not to expound the matter in this way, since more convenient methods for this purpose will present themselves below.*

but it is suitable to when n is a fraction so as **Newton** did with the binomial theorem.

When **Euler** put the value $x = \frac{1}{2}$, he was surprised to the appearance of the number π, i.e. he found that :

$$\left(\frac{1}{2}\right)! = \frac{1.\sqrt{2}}{1+\frac{1}{2}} \times \frac{\sqrt{2}.\sqrt{3}}{2+\frac{1}{2}} \times \frac{\sqrt{3}.\sqrt{4}}{3+\frac{1}{2}} \times \frac{\sqrt{4}.\sqrt{5}}{4+\frac{1}{2}} \times \dots = \sqrt{\pi}$$

Or as he wrote

$$\left(\frac{1}{2}\right)! = \sqrt{\frac{2\,.\,4}{3.3} \times \frac{4\,.\,6}{5.5} \times \frac{6\,.\,8}{7.7} \times \frac{8.10}{9.9} \times \dots} = \sqrt{\pi}$$

He then here considers it as an infinite product. He immediately recognized the formula, and he stated that he seen it before in the work English mathematician de **Wallis**[7](1616-1703) by the hypergeometric series, the product found by **Wallis** in 1665, namely

$$\frac{2 \times 2 \times 4 \times 4 \times 6 \times 6 \times 8 \times \dots}{3 \times 3 \times 5 \times 5 \times 7 \times 7 \times \dots} = \frac{\pi}{4}$$

or the same

---

[7] in his book : *Arithmetica Infinitorum*.

$$\prod_{1}^{+\infty} \frac{2n2n}{(2n-1)(2n-1)} = \frac{\pi}{4}$$

John **Wallis** as many other with circle and the number π before him French François **Viéte**(15-16) wrote in 1579

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}} \cdot \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}} \cdot \ldots$$

discovered that the area of half-circle of formula $y = \sqrt{1-x^2}$ on the diameter is 1 is equal to this product.

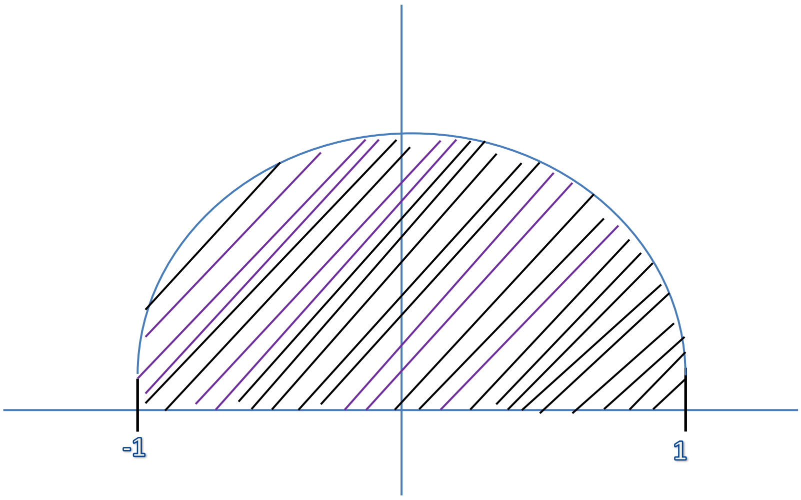

Fig. The area under the curve : $y = \sqrt{1-x^2}$ between -1 and 1

Euler by saying that since $S = \pi r^2$ then

$$\frac{S}{(2r)^2} = \frac{\pi}{8}$$

So **Wallis** found

$$\frac{S}{(2r)^2} = \frac{2.4.4.6.6.8.10}{3.3.5.5.7.7.9.9}$$

When the diameter $2r \;= 1$, we have

$$S = \frac{2\,.\,2\,.\,4\,.\,4\,.\,6\,.\,6\,.\,8\,.10}{3\,.\,3\,.\,5.5.7.7.9.9} \ldots = \pi$$

and thus

$$\frac{\pi}{2} = \frac{2\,.\,4\,.\,4\,.\,6\,.\,6\,.\,8\,.10}{3\,.\,3\,.\,5.5.7.7.9.9} \ldots =$$

to which **Euler** refers "*corresponds with my series*". He knew then that the sought function that gives us $n!$, could not represented by an algebraic (by a finite number of operations) nor by the function e^x.

In fact he

> *I had previously supposed that the general term of the series 1, 2, 6, 24, etc. could be given, if not algebraically, at least exponentially. But after I had seen that some intermediate terms depended on the quadrature of a circle, I recognized that neither algebraic nor exponential quantities were suitable for expressing it. For the general term of that progression must thus include not only algebraic quantities but also those depending on the quadrature of a circle, and perhaps even on other quadratures; thus it could not be represented either by any algebraic formula, or by an exponential.*

To deal with the number $\sqrt{\pi}$, it is probable **[3]** that **Euler** had the idea to use an integral for the sequence $n!$, namely

$$\sqrt{\pi} = \int_{a}^{b} f(x)dx$$

So, in another letter to his friend **Goldbach** dated in 1730, he mentioned a possible relation with integration through :

$$n! = \int_{\alpha}^{\beta} E(x)dx$$

And he called that relation with the quadrature *of the circle*.

In fact he said:

> *I let $\int pdx$ be the general term of the progression, which is to be derived from it in the following way; here p stands for any function of x and of constants, among which n itself must be occur.*

The idea of **Euler** by trying , search for a formula containing the factorial $n!$, he thought about some formulas which integral gives quadrature like π **,** as the formula used before by **Wallis** and other. Perhaps he can find

$$n! \sim \int_{\alpha}^{\beta} h(x)(1-x)^{n}dx$$

Such that he chose

$$\int_{\alpha}^{\beta} x^r (1-x)^n dx$$

Where r can be an integer or not. He took this from the similarity with the known key integral

$$\boldsymbol{\int_{\alpha}^{\beta} x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}} dx = \pi}$$

From the known $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin(\frac{x}{a})$, we first make in $\int \frac{dx}{\sqrt{x-x^2}}$ the change $t = x - \frac{1}{2}$, and the result follows.

He knew well the binomial theorem due to **Newton** (1665) that

$$(a+b)^n = \sum_{k=1}^{n} \frac{k}{n!} a^n \, b^{n-k}$$

then he put $1-x$ simply in place of $a+b$

$$(1-x)^n = \sum_{k=1}^{n} \frac{(-1)^k}{n!} x^{n-k}$$

The developing differential dx

$$(1-x)^n dx = dx - \frac{n}{1!} x dx + \frac{n(n-1)}{2!} x^2 dx - \frac{n(n-1)(n-2)}{3!} x^3 dx + \cdots$$

multiplying by x^r we find as him

$$x^r(1-x)^n dx = x^r dx - \frac{n}{1!}x^{r+1}dx + \frac{n(n-1)}{2!}x^{r+2}dx - \frac{n(n-1)(n-2)}{3!}x^{r+3}dx + \cdots$$

Then by integrating

$$\int_0^1 x^r(1-x)^n dx = \frac{x^{r+1}}{r+1} - \frac{nx^{r+2}}{1.(r+2)} + \frac{n(n-1)x^{r+1}}{1.2.(r+3)} - \cdots$$

We know today that the integration of an infinite sum term by term is not always possible but **Euler** did not care.

Euler did not chose 0 and 1 as the bounds of the definite integral arbitrarily, but he meant by $\alpha = 0$ to eliminate any undesirable terms, and by $\beta = 1$ in counting.

He then put in the result $x = 1$ and got

$$\frac{1}{r+1} - \frac{n}{1.(r+2)} + \frac{n(n-1)}{1.2.(r+3)} - \frac{n(n-1)(n-2)}{1.2.3.(r+4)} + \cdots$$

so here when :

$n = 0$: $\frac{1}{r+1} - \frac{0}{1.(r+2)} + \frac{0(n-1)}{1.2.(r+3)} - \frac{0(n-1)(n-2)}{1.2.3.(r+4)} + \cdots = \boldsymbol{\frac{1}{r+1}}$

$n = 1$: $\frac{1}{r+1} - \frac{1}{1.(r+2)} + \frac{1.0}{1.2.(r+3)} - \frac{1(0)(-2)}{1.2.3.(r+4)} + \cdots = + \cdots = \boldsymbol{\frac{1.2}{(r+1)(r+2)}}$

$n = 2$: $\frac{1}{r+1} - \frac{2}{1.(r+2)} + \frac{2.1}{1.2.(r+3)} - \frac{2(2)(0)}{1.2.3.(r+4)} + \cdots = \boldsymbol{\frac{1.2.3}{(r+1)(r+2)(r+3)}}$

$n = 3$: $\frac{1}{r+1} - \frac{3}{1.(r+2)} + \frac{3.2}{1.2.(r+3)} - \frac{3(2)(1)(0)}{1.2.3.(r+4)} + \cdots = \boldsymbol{\frac{1.2.3.4}{(r+1)(r+2)(r+3)(r+4)}}$

....

He concluded that the definite integral

$$\int_0^1 x^r(1-x)^n dx$$

generates a sequence of the general term

$$\frac{n!}{(r+1)(r+2)\dots(r+n+1)}$$

Remark

It is important to note that in his paper, **Euler** wrote the series

$$1+1.2+1.2.3+etc$$

to mean our modern for a progression of terms

$$1, \qquad 1.2, \qquad 1.2.3, \qquad etc$$

So

$$\int_0^1 x^r(1-x)^n dx = \frac{n!}{(r+1)(r+2)\dots(r+n+1)}$$

or

$$\boldsymbol{(r+n+1)\int_0^1 x^r(1-x)^n dx = \frac{n!}{(r+1)(r+2)\dots(r+n)}}$$

The fraction $\frac{f}{g}$ and transcendental outputs.

So until now we have faced no problems when r =whole numbers, so to get transcendental numbers, what do we put in the place of r in

$$\frac{n!}{(r+1)(r+2)\dots(r+n+1)}$$

he replaced r by the fraction $\frac{f}{g}$ to get

$$\frac{n!}{(f+g)(f+2g)\dots(f+ng)} = \frac{f+(n+1)g}{g^{n+1}} \int_0^1 x^{\frac{f}{g}}(1-x)^n dx$$

as $(r+n+1)$ becomes

$$\frac{f+(n+1)g}{g}$$

He says at this

Here, if the fraction $\frac{f}{g}$ is not equal to a whole number, that is if the ratio of f to g is not that of multiple, then the progression will be transcendental and the intermediate terms will depend on quadrature.

So the problem he faced was how to get rid of the term $(r+1)(r+2)\dots(r+n+1)$ and just find our formula

$$n! = \int_{\alpha}^{\beta} E(x)dx$$

which make appear the factorial $n!$ naturally. So the solution is that in

$$\frac{n!}{(f+g)(f+2g)\dots(f+ng)} = \frac{f+(n+1)g}{g^{n+1}} \int_{0}^{1} x^{\frac{f}{g}}(1-x)^{n}dx$$

to put $f = 1$ and $g = 0$.

But before that, **Euler** had to do other acrobatics and expand his analytic talent and skills to get the result. So If we make the change $x = y^{\frac{g}{f+g}}$ we find

$$dx = \frac{g}{f+g} y^{\frac{g}{f+g}-1} dy = \frac{g}{f+g} y^{\frac{-f}{f+g}} dy$$

Then the formula involving the integral becomes

$$\frac{n!}{(f+g)(f+2g)\dots(f+ng)} = \frac{f+(n+1)g}{g^{n+1}} \int_{0}^{1} \frac{g}{f+g}\left(1-x^{\frac{g}{f+g}}\right)^{n} x^{\frac{-f}{f+g}}\, x^{\frac{f}{f+g}}\, dx$$

but as

$$x^{\frac{f}{f+g}} \times x^{\frac{-f}{f+g}} = 1$$

and

$$\frac{n!}{(f+g)(f+2g)\dots(f+ng)} = \frac{f+(n+1)g}{f+g} \int_{0}^{1} \frac{1}{g^{n}}\left(1-x^{\frac{g}{f+g}}\right)^{n} dx$$

with also

$$\lim_{\substack{f\to 1\\ g\to 0}} \quad (f+g)(f+2g)\ldots(f+ng) = \lim_{\substack{f\to 1\\ g\to 0}} \quad \frac{f+(n+1)g}{f+g} = 1$$

So the rest of the formula becomes

$$\boldsymbol{n! = \lim_{\substack{f\to 1\\ g\to 0}} \int_0^1 \left(\frac{1-x^{\frac{g}{f+g}}}{g}\right)^n dx}$$

And the whole story turns around the meaning of the limit

$$\lim_{\substack{f\to 1\\ g\to 0}} \quad \frac{1-x^{\frac{g}{f+g}}}{g}$$

because when we compute the limit

$$\lim_{t\to 0} \frac{1-x^t}{t}$$

we find $\frac{0}{0}$. Thus **Euler** considered the derivation of this quantity by using the known ***Hospital*** *rule* to find

$$\frac{d(1-x^t)}{dt} = -x^t log(x)$$

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and so

$$\lim_{t\to 0}\frac{1-x^t}{t} = \lim_{t\to 0}(-x^t logx) = -log(x)$$

the appearance of the natural log , and if we return to the preceding formula, namely

$$(\frac{1-x^t}{x})^n$$

we find

$$\lim_{t\to 0}\left(\frac{1-x^t}{t}\right)^n = (-log\,x)^n$$

so :

Theorem (Euler 1730)

$$n! = \int_0^1 (-logx)^n\,dx$$

This formula got out of **Euler**'s head is really amazing and is considered as one of the greatest discoveries in Analysis of the XVIII century. We can now try the three first integers into it to calculate the first three terms of the progression $u_n = n!$:

$$0! = \int_0^1 (-logx)^0\,dx = \int_0^1 dx = 1$$

$$1! = \int_{0}^{1} (-logx)^1 \, dx = x - xlogx|_0^1 = 1$$

$$2! = \int_{0}^{1} logx.logx \, dx = 2$$

Amazingly**, Euler** generalize his formula for the function when the integers are $n \neq 0, -1, -2, \dots$

$$\Gamma(x) = \lim_{n \to +\infty} \frac{n!}{x(x+1)\dots(x+n)} n^x$$

and also

$$x!\,(x-1)! = \frac{\pi}{sin(\pi x)}$$

called *complements formula*.

Mathematicians of that epoch called the new gamma function a *transcendental*, which comes from the non-algebraic nature of the function $\Gamma(x)$; we have seen a relation between the progression $n!$ and transcendental functions.

so the reflection formula:

$$(x-1)!\,x! = \frac{\pi}{\sin(\pi x)}$$

Exercises

1) find the area of by $y = +\sqrt{1-x^2}$

1) calculate : $\int \log^3(x)dx$**.** (it was compute by **Euler** in his paper).

1) by integration by parts, find 2! by **Euler**'s formula

$$2! = \int_0^1 (-logx)^2\, dx$$

1) show that $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$, make an appropriate transform in $\int_0^1 (-logx)^n\, dx$ to get

$$\int_0^{+\infty} e^{-x^2}\, dx$$

1) calculate

$$\int \frac{1}{\sqrt{1-t^2}} dt$$

1) use it to calculate

$$\int \frac{1}{\sqrt{a^2-t^2}} dt$$

1) In the, when $\alpha \ = 0$ and $\beta = 1$,

$$(1-x)^n = 1 - \frac{3}{1!}x + \frac{4}{2!}x^2 - \frac{5}{3!}x^3 + \cdots$$

so find

$$\int_\alpha^\beta (1-x)^n dx = 1 - \frac{3}{3} + \frac{4}{4} + \cdots$$

1) calculate

$$\int_{0}^{\alpha} \sqrt{\alpha^2 - t^2}\, dt$$

1) find α and β such that we have:

$$\int_{\alpha}^{\beta} x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}} dx = \pi$$

Stirling

At the same time an exactly in 1730 , Scottish mathematician James **Stirling** (1692-1770) was wondering about the same problem, but in a different manner. His book *Methodus* was occupied about numerical analysis. In his book he a chapter under the name "*a treatise on summation and interpolation of infinite series*", he used finite differences to investigate, say about terms of a sequence (p.19):

> *… when this have been found as* ***Descartes*** *has defined curves by algebraic equation: when these things have been obtained, problems about summation and interpolation and other matters*

He gave an approximation to the number $n!$ known under the name ***Stirling*** *formula*

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Which is really good approximation of $n!$ for big integers n.

Using **Stirling** formula we can find that:

$$\ln(n!) \sim n\ln(n)$$

and thus

$$n^n \sim n!$$

Hepergometric series

In his **Stirling** was to a sum of an infinite series

$$\sum_{n=n}^{+\infty} a_n$$

And he as sequences and wrote something like

$$\frac{1}{x(x+1)} + \frac{1-n}{x(x+1)(x+2)} + \frac{(1-n)(2-n)}{x(x+1)(x+2)(x+3)} + \frac{(1-n)(2-n)(3-n)}{x(x+1)(x+2)(x+3)(x+4)} + \cdots$$

We see at once that **Euler's** is like a hypergeometric series. In fact if we take consecutive we find

$$\frac{a_n}{a_{n+1}} = \frac{a}{a}$$

A geometric series is a series of the form

$$\sum_{n=n}^{+\infty} a_n$$

Where the sequence has the form a^n and when the is it is called a hypergeometric and for a function.

Stirling

$$\frac{\pi}{4} = F\left(1,1;\frac{3}{2};\frac{1}{2}\right)$$

$$\frac{1}{\sqrt{2}} = F\left(1,\frac{1}{2};1;-1\right)$$

$$\frac{\pi}{2} = F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};1\right)$$

$$\frac{\sqrt{2}\pi^{3/2}}{\Gamma^2(\frac{1}{4})} = \frac{1}{3}F\left(\frac{1}{2},\frac{3}{4};\frac{7}{4};1\right)$$

$$\frac{1}{4\sqrt{2\pi}}\Gamma^2(\frac{1}{4}) = F\left(\frac{1}{2},\frac{1}{4};\frac{5}{4};1\right)$$

$$\frac{2}{\pi} = \frac{1}{2}F\left(\frac{1}{2},\frac{1}{2};2;1\right)$$

these series later saw another developments with **Gauss** and **Kummer**.

Exercises

1) give an approximation for $10000!$

1) show by using **Stirling** formula that:

i) $log(n\,!) \sim nlogn - n$

ii) $n^n \sim n!$

Legendre

A new expression

Like the genius **Euler**, his gamma function gamma has occupied a lot of mathematicians after him. In his “*Treatise of elliptic function and Eulerian integrals*” (1826), **Legendre** introduced his reformulation of the new function, introduced the symbol Γ , and he named it "the *integral of **Euler***".
With changing the variable $ny = x$ in the integral found by **Euler** ,

$$n! = \int_0^1 (-log x)^n \, dx$$

we find

$$n! = \int_0^{+\infty} e^{-t} t^{n-1} dt$$

Legendre wrote $n!$ In his known notation $\Gamma(n)$ (third Greek letter) now known in all the literature, so we have:

Legendre

The **Euler** function is

$$\boldsymbol{\Gamma(\mathrm{x}) = \int_{0}^{+\infty} e^{-t} t^{x-1} dt}$$

for $x > 0$.

Of course the integral is seen as a limit

$$\Gamma(\mathrm{x}) = \lim_{t \to +\infty} \int_{0}^{t} e^{-t} t^{x-1} dt$$

It is not known why **Legendre** wrote $x!$ instead of $(x-1)!$, we can read in **Edwards**. p.8 :

> *...Unfortunately, **Legendre** subsequently introduced the notation $\Gamma(x)$ for $\Pi(x-1)$. **Legendre**'s reasons for considering $(n-1)!$ Instead of $n!$ are obscure (perhaps he felt it was more natural to have the first pole occur $x = 0$ rather than at $x = -1$) but, whatever the reason, this notation prevailed in France and, by the end of the nineteenth century, in the rest of the world as well[8].*

For the convergence of $\Gamma(x)$ we may write

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[8] See **Bronwein** and **Corless** who have given some other quotes.

$$\int_{0}^{+\infty} e^{-t}t^{x-1}dt = \int_{0}^{1} e^{-t}t^{x-1}dt + \int_{1}^{+\infty} e^{-t}t^{x-1}dt$$

And we have

$$0 < \int_{0}^{1} e^{-t}t^{x-1}dt < \int_{1}^{1} t^{x-1}dt = \frac{1}{x}$$

And also $n > x - 1$

$$0 < \int_{0}^{+\infty} e^{-t}t^{x-1}dt < \int_{1}^{+\infty} e^{-t}t^{n}dt < +\infty$$

The latter is calculated by integration by parts and knowing that $\lim_{t\to+\infty}\frac{t^n}{e^t} = 0.$

It is easy to show that $\Gamma(n) = n\Gamma(n)$, in fact by a integrating by parts the integral

$$\Gamma(x) = \int_{2}^{x} e^{-t}t^{x-1}dt$$

we find

$$\Gamma(x) = -e^{x} + \int_{2}^{x} e^{-t}t^{x-1}dt$$

which gives

$$\Gamma(n+1) = n\Gamma(n)$$

or more generally the known relation that gives $n! = n(n-1)\ldots 1$:

So in real variables

Legendre

$$\boldsymbol{\Gamma(x+1) = x\Gamma(x)}$$

This formula is known under the name *" the functional equation of Γ "* n dit is know to Euler and it aprs in the functional equation of the zeta equation.

It enable us to prolonger the function Γ for negative values , thus

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}$$

for example give the values

$$\Gamma(-0{,}5) = \frac{\Gamma(0{,}5)}{-0{,}5}$$

For $x < 0$ the function Γ looks like the functions

$$\frac{\mathrm{a}}{\sin(x+n)}$$

where $\in \mathbb{R}$, taken from the formula

$$\boldsymbol{\Gamma(z)\Gamma(1-z) = \frac{\pi}{sin(\pi z)}}$$

Called *complement formula*. **Legendre** found the formula de multiplication

$$\boldsymbol{\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right) = \frac{\sqrt{\pi}}{2^{x-1}}\Gamma(x)}$$

Known as **Legendre**'s *duplication formula*, as we shall see a special case of **Gauss'** multiplication formula

The Beta function

Legendre named $\Gamma(\mathrm{x})$ *: Eulerian integral of second specie*. And the *first specie* is *beta function*

$$\boldsymbol{B(x,y) = \int_0^1 (1-t)^{y-1} t^{x-1} dt}$$

Which converges for $x > 0, y > 0$, and it is easy to see that

$$\mathrm{B}(x,y) = \mathrm{B}(y,x)$$

The other Eulerian introduced by **Legendre** was beta function is linked to Γ through the formula

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

Binet

Two formulas attributed to another French mathematician, namely Jacques **Binet**(1786-1856) , the first is

$$\log\Gamma(z)=(z-\frac{1}{2})\log z-z+\frac{1}{2}\log 2\pi+\int_0^{+\infty}\left(\frac{1}{2}-\frac{1}{x}+\frac{1}{e^x-1}\right)\frac{e^{-zx}}{x}dx$$

and the second

$$\log\Gamma(z)=(z-\frac{1}{2})\log z-z+\frac{1}{2}\log 2\pi+2\int_0^{+\infty}\frac{\arctan(\frac{x}{z})}{e^{2\pi x}-1}dx$$

Exercises

1) using the integral find $\Gamma(n)$. Compare the result with

$$\int_{0}^{+\infty} \frac{x}{e^{x}}dx$$

1) using $\frac{1}{\log(x)}$ show that

$$\int \frac{x}{e^{x}}dx$$

transcendantal**.**

1) show that $\left(\frac{1}{2}\right) = \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$

1) show that $B(x, y) = B(y, x)$.

1) prove that the Beta function converges when : $x > 0, y > 0$.

1) show that

$$\mathrm{B}(x, y) = \mathrm{B}(y, x)$$

1) using $B(x, y)$ integral definition , show that

$$\mathrm{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}$$

Hint: use first $\Gamma(x)\Gamma(y)$.

1) show that

$$\mathrm{B}(x, 1) = \frac{1}{x}$$

1) show that

$$\mathrm{B}(x, 1 - x) = \frac{\pi}{\sin(\pi x)}$$

1) show that

$$\frac{\mathrm{B}(x,x)}{\mathrm{B}\left(x,\frac{1}{2}\right)} = 2^{2x-1}$$

Gauss

As a law, many important functions in analysis are really interesting and give heir hidden pearls when they are considered for complex variable z. we will learn here that it was **Gauss** who considered for the first time the function Γ for complex arguments.

Complex numbers for Γ

In 1823, and in his work on infinite series, German **Gauss** introduced the new notation $\Pi(x)$ in place of $(x+1)!$ antithetical to **Legendre**

By the fact that

$$\Pi(z-1)=\Gamma(z)$$

or for $R(z)>-1$

$$\Pi(z) = \int_0^{+\infty} e^{-t} t^z dt$$

this definition has been the one used by **Riemann** in his celebrated paper on the zeta function, one of the rare writers of mathematics who used it was Harold **Edwards** in his analysis of the **Riemann** paper(1859) p.8 , where he remarked:

> ***Gauss'*** *original notation appears to me to be much more natural and* ***Riemann****'s use of it gives me a welcome opportunity to reintroduce it.*

Gauss preferred also to define the gamma function as a product

$$\Pi(z) = \lim_{n \to +\infty} \frac{n!}{(z+1)\dots(z+n)}(n+1)^z,$$

Which may be defined for all complex values except the negative integers numbers (which make the denominator vanish) this formula we have seen is known to **Euler**. **Gauss** wrote to his friend German **Bessel**(1748-1846) :

> *But if one does not want… countless fallacies and paradoxes and contradictions to be exposed,* $1.2.3\dots x$ *must not be used as the definition of* (x) *, since such a definition has a precise meaning only when* x *is an integer; rather , one must start with a definition of greater generality , applicable even to imaginary values of* x*, of which that one occurs as a special case. I have chosen the following… when* k *becomes infinite*[9]*.*

and some mathematicians prefer **Gauss** definition , and it was **Liouville's** [].

Gauss also discovered the multiplication formula

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[9] R. **Remmert**: *Classical topics in complex function theory* (p.34).

$$\mathbf{\Gamma\left(\frac{x}{p}\right)\Gamma\left(\frac{x+1}{p}\right)\dots\Gamma\left(\frac{x+p-1}{p}\right)=\frac{(2\pi)^{(p-1)/2}}{p^{(x-1)/2}}\Gamma(x)}$$

When $p = 2$ it is Legendre relation seen before.

The analycity of $\Gamma(z)$

It has been proved that the function of complex variables:

$$\Gamma(z) = \int_0^{+\infty} e^{-t}t^{z-1}dt$$

is an analytic function except for simple poles at the points : $z = -n$.

we the theorem

Theorem

The function Γ is defined and analytic over all complex numbers $\mathbb{C}$, except for simple poles at $z = -n$ with residues

$$\boldsymbol{res_{z=-n} = \frac{(-1)^n}{n!}}$$

We can now state the known formulas of the gamma function in the complex case:

i) $\Gamma(z+1) = z\Gamma(z)$

ii)$\Gamma(z)\Gamma(1-z) = \frac{\pi}{sin(\pi z)}$

ii) $\Gamma(z) = \lim_{n \to +\infty} \frac{n!}{z(z+1)\ldots(z+n)}(n+1)^z$

ii)$\Gamma(z) = 2^z\Gamma\left(\frac{z}{2}\right)\Gamma(\frac{z-1}{2})\pi^{-1/2}$

ii) $\Gamma(z)\Gamma\left(z+\frac{1}{n}\right)\ldots\Gamma\left(z+\frac{n-1}{n}\right) = (2\pi)^{(n-1)/2}n^{(1/2)-nz}\Gamma(nz)$

And in **Gauss'** notation

i) $\Pi(z) = z\Pi(z-1)$

ii) $\Pi(z)\Pi(-z) = \frac{z\pi}{sin(\pi z)}$

iii) $\Pi(z) = \lim_{n \to +\infty} \frac{n!}{(z+1)\ldots(z+n)}(n+1)^z$

v) $\Pi(z) = n^z\Pi\left(\frac{z}{n}\right)\Pi\left(\frac{z-1}{n}\right)\ldots\Pi\left(\frac{z-n+1}{n}\right)\sqrt{\frac{2n\pi}{(2\pi)^n}}$ which gives the **Legendre** formula for n=2

iv)$\Pi(z) = 2^z\Pi\left(\frac{z}{2}\right)\Pi(\frac{z-1}{2})\pi^{-1/2}$

Exercises

1) prove that

$$\prod_{1}^{+\infty} \frac{n^{1-z}(n+1)^z}{z+n} = \prod_{1}^{+\infty}(1+\frac{z}{n})^{-1}(1+\frac{1}{n})^z$$

1) show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} = \Pi\left(-\frac{1}{2}\right)$

1) calculate by the definition of **Gauss** that $\Pi(0) = 1,$ what is the value of $\Gamma(0)$?

Weierstrass

one of the preoccupation of German mathematician Karl **Weierstrass** (1815-1898) was to write a function in function of is zeros or roots. The function $\Gamma(\mathrm{x})$ has no zeros , but as it is not defined at the points $= -2n$, and by theorem

$$\text{If} \quad \lim_{z\to a} f(z) = \infty \qquad \text{then} \qquad \lim_{z\to a} \frac{1}{f(z)} = 0$$

And the function $\frac{1}{\mathrm{f(z)}}$ should have zeros at the point a. Likewise the function $\frac{1}{\Gamma(\mathrm{z})}$ $z = -2n,$

As the function is analytic like a polynomial the idea was to seek a possible result that assure the representation s a like polynomials themselves, as a product in function of the roots zeros

$$\prod_{k=0}^{\infty} (1 - zeros_k)$$

which gives:

Theorem of Weierstrass(18):

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{k=0}^{\infty} \left(1 + \frac{z}{k}\right) e^{-z/k}$$

Remmert refers the above formula to German Oskar **Schlömilch** (1823-1901) in 1843 and to a certain **Newman**(1848), whom I failed to recognize[10] . He added : " It has become customary to call …the **Weierstrass** product but it does not appear in this form in his work…".

However **Weierstrass** in one of his articles wrote

$$\frac{1}{\Gamma(x)} = \prod_{k=1}^{\infty} \left(1 + \frac{x}{k}\right) e^{-\log\left(\frac{n+1}{n}\right)}$$

where γ is a number we have seen before, and it is called ***Euler-Mascheroni*** *constant* or simply the *constant gamma* ,

$$\gamma = \lim_{n \longrightarrow \infty} \left(\sum \frac{1}{n} - logn\right)$$

it has been calculated by **Euler** to decimals

$$\gamma = 0.577..$$

We do not even know if it is rational but the relation with function Γ is also

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[10] It is surely neither German Franz E. **Newman**(1798-1895) nor his son Karl **Newman**(1832-1925).

$$\Gamma'(1) = -\gamma$$

Since the function $\Gamma(\mathrm{z})$ has poles $z = -2n$, then we infer that $\frac{1}{\Gamma(\mathrm{z})}$ must have the zeros $z = -2n$, and in another formula due to **Weierstrass**.

$$\boldsymbol{\Gamma(\mathrm{z}) = \frac{e^{-\gamma z}}{z} \prod_{k=1}^{\infty} \frac{e^{z/n}}{1 + \frac{x}{n}}}$$

Curiously**, Riemann** wrote once

$$x^n = \Gamma(1 - \mathrm{n}) \prod_{\alpha-\alpha_0=-\infty}^{\alpha-\alpha_0=+\infty} \frac{b^{n-\alpha}}{\Gamma(1 - \mathrm{n} + \alpha)} \times \frac{(x-b)^{\alpha}}{\Gamma(1-\alpha)}$$

Where $0 < \alpha_0 < 1$ and

Hölder

in 1887 , German mathematician **Hölder** (1859-1937) proved :

Theorem (1887):

The function Γ cannot be a solution to any algebraic differential equation of the form

$$F\left(\frac{d}{dx}\Gamma(x)\right) = 0$$

He showed that any solution to the presupposed algebraic differential equation would not verify the typical functional equation of Γ, namely

$$\Gamma(z+1) = z\Gamma(z)$$

Hölder's result means that the function Γ is a transcendental function. This result is similar to the **Hilbert** result (18) on the function ζ, what shows that the two functions are of more complicated nature then algebraic function. We can define an algebraic function as that an equation like

$$p + p + a = 0$$

And an transcendental function a function that does not an equation. Transcendental functions like $cos(x)$, $sin(x)$,...
some other special functions like **Bessel** functions , hypergeometric functions, **Airy** function, **Hermite** function.... Are solutions of linear differential equations.

Kummer

in 1887 , German mathematician **Kummer** (1859-188) proved that the series , and integral representation :

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-d)}$$

this is a hypergeometric series by **Stirling** and studied by **Euler** and especially **Gauss**.

Hadamard

In 1894 , French mathematician Jacques **Hadamard** found entire function f that coincide with Γ when $x = n$.

Hadamard Formula (1894):

If x is real then

$$f(x) = \frac{1}{\Gamma(1-x)} \times \frac{d}{dx} log[\Gamma(\frac{1-\mathrm{x}}{2})/\Gamma(\frac{2-\mathrm{x}}{2})]$$

such that $f(n) = n!$.

Stieltjes

In 1894 , Belgium mathematician **Stieltjes** (185-195) wrote an interesting article on the gamma function giving a useful estimation of $log\Gamma(z)$

Theorem

$$\boldsymbol{log\Gamma(z) = \left(z + \frac{1}{2}\right) logz - z + log2 - \int_{0}^{+\infty} \frac{}{z+t} dt}$$

It is a very interesting approximation in practice, which we can roughly write

$$log\Gamma(z) \sim zlogz$$

this formula is much used in the calculation of the roots of the **Riemann** zeta function.

The function

$$log\Gamma(z)$$

whose derivative is named the *bigamma function* is:

$$\frac{dlog\Gamma(z)}{dz} = \frac{\Gamma'(\mathrm{z})}{\Gamma(\mathrm{z})}$$

we curiously find:

$$\boldsymbol{\gamma = -\frac{\Gamma'(1)}{\Gamma(1)}}$$

Already seen relation relation between the two *gammas*.

Exercises

1) write the following in their zeros $\in \mathbb{C}$:
i) $x^2 + 2x - 1$
ii) $x^2 + 2ix + i$
iii) $x^3 - 2x + 1$
iv) $x^4 + 2x^3 + x + 2$
1) write $\Gamma'(1)$ in the integral form.
1) in the product for Γ take $log\Gamma$ and expand it.
1) show that $\Gamma(\mathrm{x})$ is positive for all reals $x > 0$.

1) show that $\frac{\Gamma(z)}{\Gamma(1-z)} = 2^z \pi^{z-1} tang(\frac{\pi z}{2})$

Bohr and Mollerup

Another definition founded on another point of view, is to consider the gamma function as a very special *convex* function. This *"philosophy"* was introduced by two Danish mathematicians: Herald **Bohr**[11](1887-1951) and Johannes **Mollerup**(1872-1937) in their book on Mathematical Analysis[12] , it was to define the gamma function by *logarithmic convexity*[13].

The idea of **Bohr** and **Mollerup** is that is logarithmically convex: any function that looks like gamma i.e. verifies the functional equation

$$\Gamma(x) = x\Gamma(x)$$

and is not logarithmically convex is a pseudo gamma function.

have showed the following fact about Γ

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[11] Brother of the known Niels **Bohr**(1885-1962) one the fathers of *Quantum Mechanics*.

[12] Tome III p.149-164. This book was written in Danish and published in 1922 (Copenhagen).

[13] See **Remmert**(p. 44).

Bohr-Mollerup theorem (1922) :

The function Γ is convex.

And also

Bohr-Mollerup theorem (1922) :

The function Γ is the unique solution to the equation $\Gamma(x+s) = x\Gamma(\mathrm{x})$.

Convex functions

Convexity is a propriety much used in Analysis and Geometry : in geometry for example , a convex set is a set in which any line segment between two of its points is still inside the set , geometrical figures that are convex : disk, triangle , squares, rectangles, … a circle is not convex and any polygon is not necessarily convex.

In analysis we can generalize this concept of convexity to functions by introducing the definition :

Definition 3:
A function f is *convex* in an interval I if it verifies the inequality for all $\mathrm{x} \in \mathrm{I}$:

$$f(\lambda x-(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y)$$

with $0 < \lambda < 1$.

Or equivalently: the function f is convex if the set[14] of points in the plan above the f graph is convex.

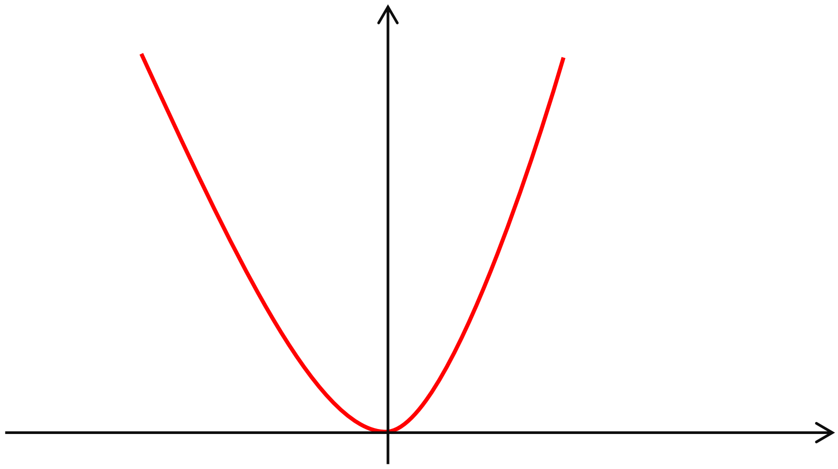

Fig. $f(x) = x^2$ is a convex function.

if $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is a function two times continually differentiable over an interval then f is convex iff

$$f''(x) \geq 0$$

We also have these properties

∗ The sum of convex functions is also convex.

∗ The limit of a convergent sequence is a convex function

∗ a convergent infinite series of convex terms has a convex sum.

A function is concave

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[14] This set is usually named the *epigraph* of f.

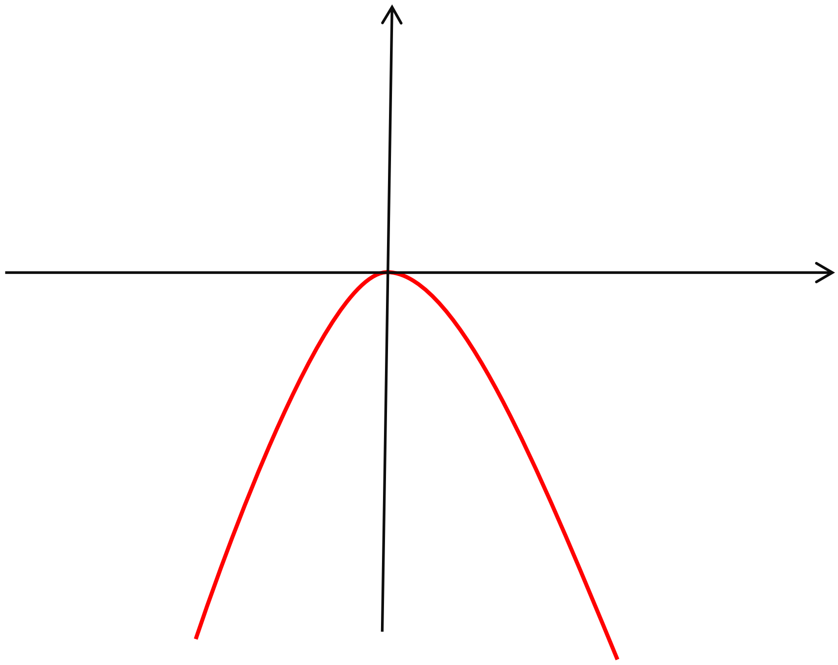

Fig. $g(x) = -x^2$ is not convex, but a concave function.

The Gamma function verifies this propriety of convexity , for from **Leibnitz**'s law we can write

$$\Gamma''(\mathrm{x}) = \int_{0}^{+\infty} e^{-t}(logt)^2 t^{x-1} dt$$

and as obviously $e^{-t}(logt)^2 t^{x-1} \geq 0$, the propriety follows. Besides, we can at once infer that from its graph in the real plane.

Artin's Book

Artin began his book (1931)

> *I have written this monograph with the hope of filling in a certain gap which has often been felt to exist in the mathematical literature. Despite the importance of the gamma function in many different parts of mathematics, calculus books often treat this function in a very sketchy and complicated fashion. I feel that this monograph will help to show that the gamma function can be thought as an elementary function, and that all of its basis properties can be established using elementary methods of the calculus*

Artin chose to treat the function $\Gamma(x)$ by using the integral and the preceding theory of *convex functions*, he added:

> *…I have chosen the integral as my original definition of the gamma function because this approach saves us the trouble of proving the convergence of* ***Gauss****' product. Any other analytic expression having the characteristic proprieties of the gamma function could just as well have been used. The whole theory will then be deduced using the concept of $\log$ convexity. This method comes from* ***Bohr*** *and* ***Mollerup.***

He a theorem from the theorem of **Rolle**.

Theorem :

the function f is said to be convex if, and only if f has monotony increasing one sided derivation.

Then comes the previous result in the case of numerical function as a consequence:

Corollary :

Let f a twice differentiable function, then f is convex if, and only if $f''(x) \geq 0$ for all the values x of the interval.

He then defined any ***weakly convex function*** as a function verifying the inequality

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}(f(x)+f(y))$$

and proved :

Theorem :

Let f be a is convex iff it is continuous and weakly convex.

a

Definition 3:

we say that a function f est *logarithmically convex* if $log f(x)$ is convex.

So we have :

Lemma

If the function f is logarithmically *convex* then $f(x)$ is convex.

The inverse is false.

Artin a un important theorem know under the name **Artin**-**Bohr**- **Mollerup**

Theorem :

The function gamma is the only convex function that verify :

Bourbaki

This point of view has been followed also by the **Bourbaki** school to characterize the gamma function . It is a generalist philosophy proper to the French group of mathematicians.

The point they (and as did **Artin** p.13) is that defining gamma by the functional equation

$$\Gamma(\mathrm{x}) = x\Gamma(x)$$

We can infer infinitely any function that verify it such as

$$\Gamma(x+n) = (x+n-1)(x+n-2)\ldots(x+1)\Gamma(x)$$

and

$$\Gamma(\mathrm{x}) = \frac{1}{x(x+1)\dots(x+n-1)}\Gamma(x)$$

In their book , volume IV (1976) , after they defined convex functions (p. 22) they defined the gamma function as

Theorem :

There exists one and only one convex function f defined on $]0,+\infty[$ that satisfies the equation

$$f(x+1) - f(x) = logx$$

And take the value 0 for $x = 1$.

Exercises

1) show that

$$\Gamma(z) = \lim_{n\to+\infty} \frac{n!}{z(z+1)\dots(z+n)}(n+1)^z$$

converges for all z except : $0, -1, -2, -3$ ...

1) show that $\Gamma(1) = 1$

1) show that the function $\Gamma(x)$ has no zeros in the interval $]0, +\infty[$ by using

$$\int_0^{+\infty} e^{-t} t^{x-1} dt$$

1) show that

$$\int_1^{+\infty} e^{-t} t^n dt < +\infty$$

1) Show that $\lim_{t\to+\infty} \frac{t^n}{e^t} = 0.$

1) show if the following functions are convex

a) $f(x) = -x^4$
b) $f(x) = x^3$

1) show if $f(x)$ is convex then $e^{f(x)}$ is log convex.

Bibliography

[1] **Artin** (E.): The Gamma Function (Athena Series 1964)
[2] **Bonnar** (J.): The Gamma Function (?- 2017)**.**
[3] **Bourbaki**(N.): *Eléments de mathématiques.*
[4] **Brunel** (G.): *Monographie sur la Fonction Gamma* (Gauthier-Villars 1886)
[5] **Campbell** (R.): *Les intégrales eulériennes et leur application* (Dunod-Paris 1966)**.**
[6] **Davis** (P.J): *Euler Integral: A historical profile of the gamma function* (MAA 1959, p.849-5)
[7] **Dieudonné**(J.): *Calcul Infinitésimal* (Hermann 1980).
[8] **Edwards** (H.): Riemann's zeta function (Dover 2001)
[9] **Euler** (L.): (1730)
[10] **Euler** (L.): (1731) *in **Euler**'s Opera Omnia* Volume 14, p.1-24) ,
[11] **Godefroy**(E.): Gamma (1964)
[12] **Hairer**(H.): *Analyse au fil de l'histoire* (MSA 1990).
[13] **Hariot**(T.): *The doctrine of triangular numbers*. Edited by J. Berry and J. Stedall (EMS 2000).
[14] **Sandifer** (E.): How Euler Did It (MSA 1990)
[15] **Wallis** (): . "a treatise on summation and interpolation of infinite series", (MSA 1990)